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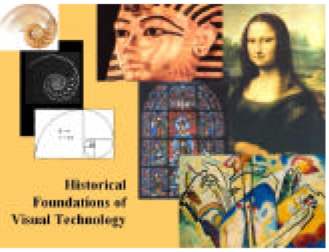


Geometry, perfect solids, golden mean

This reading was developed especially to review several definitions and concepts of geometry that relate to visual technologies and design. This required reading sets the stage for your accomplishing your homework assignment #2.

Basic Definitions

A **point** is a place in space. It is an imaginary construct that has no dimensions. You can think of it more as a location than a physical thing.

A **line** is a connection between two different points. A line has only one dimension, its length. A line has no width or height. One way to think about a line is to imagine it as a rope connecting two points, except that in this case the rope has no "thickness." Another way to think about a line is to picture it as what you would see if a luminous point moved between two other fixed points.

A **plane** is what you get if you "sweep" a line through space. If you do this in such as way as to use only two dimensions, you define a flat surface. A plane has only two dimensions: length and width. A plane has no thickness any more than a line has thickness. You can think about it as a flat sheet of paper, but as a theoretical concept a plane has no bound like a cut sheet of paper does—the plane extends outward in its two dimensions as far as the universe goes.

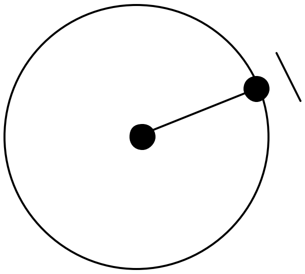
We live in a world in which we experience **three dimensions**, commonly thought of as **length**, **width**, and **depth**. As humans we are biologically and neurologically "wired" to sense our surroundings in these terms. We have two eyes and a brain that "triangulates" the observations of our two eyes. The brain integrates the slightly differing observations of the two eyes into a mental construct or visualization of three-dimension space. This lets us move through three-space with a visual sense of distance, or **depth perception**. Whether more than three dimensions exist in the universe is a matter of scientific speculation—many physicists and cosmologists postulate that many more dimensions exist, out of the range of human perception. But three dimensions are all we have to work with visually and it's all we consider here.

A **shape** is a collection of points connected with lines. For example, three points not on the same line can be connected with three lines, forming a triangle. One point swept around another point so as to remain in the same plane and always the same distance from the other point maps out a circle. Four equally space points on a circle when connected map out a square.

A **solid** is mapped if you move a two-dimension shape through three-dimension space. If the shape was a square, the movement of it in a third dimension for a distance equal to the length of a side of the square can define a cube. There are other ways to construct solids. For example, if a point exists above a circle and a line is drawn from that point to the circle and then rotated following the circle, the moving line maps out a cone. Solids are also known as **polyhedra**. There is a set of five solids that were long ago discovered to be very special, and are named **Platonic Solids**.

The Circle

A **circle** is defined by rotating one point about another point, keeping the moving point in the same plane and the same distance from the other point as it is rotated:



This is exactly what you do when you use a compass with a point on one end and a pencil on the other to draw a circle. You can create a circle using any type of material for the line connecting the fixed point (the point at the center of the circle) to the point you are sweeping around the center. In ancient times, people used rope for this line, attached to a stake at the center. The monuments and stones at Stonehenge were most likely laid out in this way.

The **radius** of a circle is the length of the line from its center to the curved line forming the circle itself. The **diameter** is twice the radius, that is, the radius line continued to the opposite side of the circle. The **circumference** of the circle is the length of line that forms the circle (the curved line). The length of the circumference in relationship to the radius has often been regarded as mysterious, especially by ancient peoples. The circumference is related to the length of the radius by this formula ("x" here is the multiplication sign):

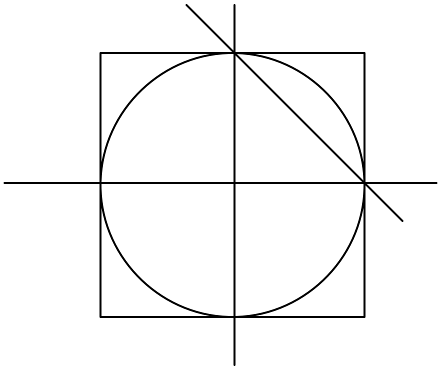
circumference = 2 x radius x pi

Pi is a number that can be computed in a variety of ways but can only be represented (in base 10 numbers) by a series of digits that begin with 3.14 and go on forever. For example, here is pi to 50 decimal places:

314159265358979323846264338327950288419716939937510…

Pi is roughly approximated by the fraction 22/7, a fact some ancient people determined and used to lay out and construct circular structures.

The area of a circle is also related to pi. Suppose you drew four squares to cover the circle, each with a side the same length as the radius:



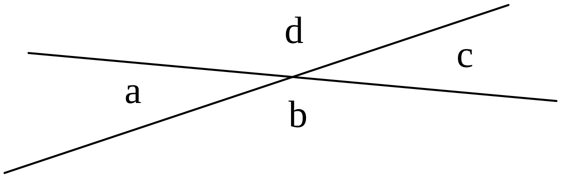
The area of each of these four squares would be

radius x radius

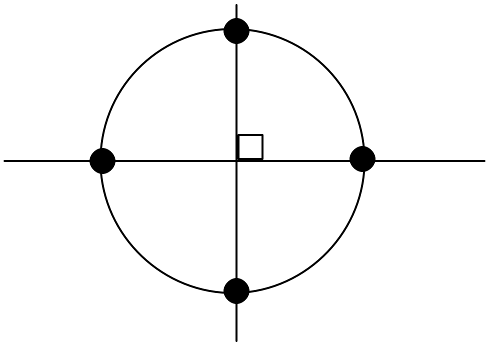
and since we have four squares the area of the square surrounding the circle would be 4 x radius x radius, which we could also write as 4 x radius^2 since the superscript 2 means "squared" (that is, multiplied by itself). The area of the circle is clearly less than this area. In looking at the area of just one-quarter of the circle, you can see that this area is more than half the area of the square with length radius, and less than whole area of the square's area, which is radius x radius. This is especially clear if you cut one of the radius x radius squares in half diagonally. It turns out that the area of the circle is pi x radius x radius, that is, pi x radius^2. The mystery deepens…

Angles

Four **angles** are formed when two lines intersect:



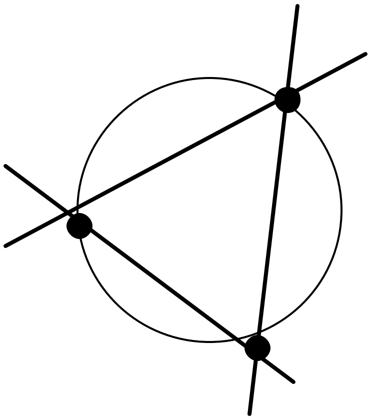
In this example two lines intersect; in doing so they have exactly one point in common. The four angles a, b, c, and d are formed. Angles a and c are equal, and angles b and d are equal. Angles are commonly measured in degrees. The sum of the angles a, b, c, and d is 360 degrees. This is an important definition because it tells us that all of the angles surrounding a point (or a circle) add up to 360 degrees. If we have a circle and place four points on its circumference as far apart as they can be from each other, they will space out evenly around the circle:



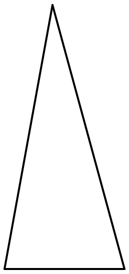
Now if we join the opposite points together, we have split the circle into four equal parts, and we have split the 360 degrees around the point at the center of the circle into four equal parts. 360 divided by 4 is 90 degrees. 90 degrees is called a **right angle,** which is usually denoted with a small box at the angle. A right angle is one corner of a square; a square encloses four of them.

Geometric Shapes

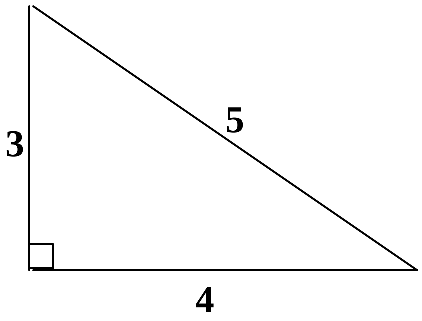
Plane (flat) shapes can be defined using a circle with points on the circumference. For example, if we place three points on the circumference of a circle, such that they are each as far apart from each other as possible, we can then connect the points to form an equilateral triangle:



All sides of the **equilateral triangle** are of equal length. It's possible to construct triangles in which the sides of not of equal length. If only two sides are of equal length, the shape is called an **isosceles triangle**:

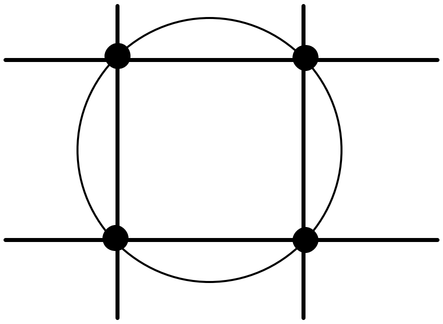


A **right triangle** has the property of having one of two of its sides forming a 90 degree angle. While an infinite number configurations of right triangles exist, a special relationship documented on a Babylonian tablet dated to 1900-1600 BCE, and described by Pythagoras (560-480 BCE) lets us construct a right triangle very easily. If we use lines of length 3, 4, and 5 to form a triangle, the angle between the sides of length 3 and 4 will be a right angle. The Egyptians used this to construct a tool named a "square" to aid in constructing things precisely vertical:

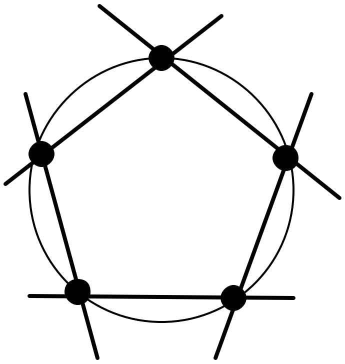


Why is this important? If you want to construct something that is square, you need to be able to compare what you are building to a right angle—hence the name of this masonry tool, the square. Similarly, unless you want a structure (like a wall or tower) to lean and become unstable, you need to construct it precisely vertical in relationship to the center of the earth. To do this you need to know what 90 degrees is, and you need to know what is "level." But level that isn't hard to figure out. The surface of any still pond of water or bowl of water is level. And a string with a weight on the end hangs vertically.

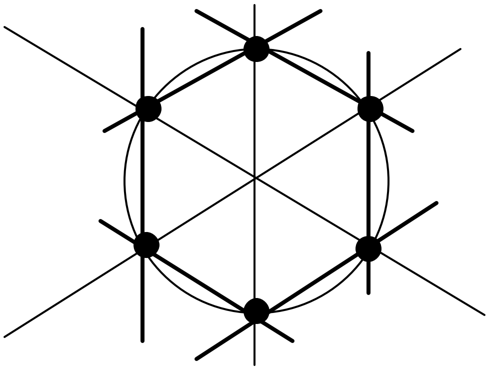
If we have a circle and place four points on its circumference as far apart as they can be from each other, they will space out evenly around the circle. If we connect the four points with straight lines the result will form a **square**. Each side of the square will be of the same length, and each corner of the square will have a 90 degree angle:



If we have a circle and place five points on its circumference as far apart as they can be from each other, they will space out evenly around the circle. If we connect the points we will form a **pentagon**:



If we have a circle and place six points on its circumference as far apart as they can be from each other, they will space out evenly around the circle. If we connect the points we will form a **hexagon**:



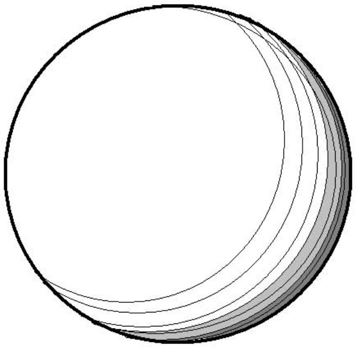
Interestingly, in the case of the hexagon, the length of each side will be exactly equal to the radius of the circle. If opposite points are connected we form six equilateral triangles.

Solids

Solids are three dimensional objects. It was discovered in ancient times that only a small set of "perfect" solids exists. We have since learned that these shapes form the buildings blocks of three-dimensional space, and a basis for many things in chemistry and atomic physics! These are named the **Platonic Solids** since they were first documented by Plato in his work ***Timeaus*** between 427 BCE and 347 BCE.

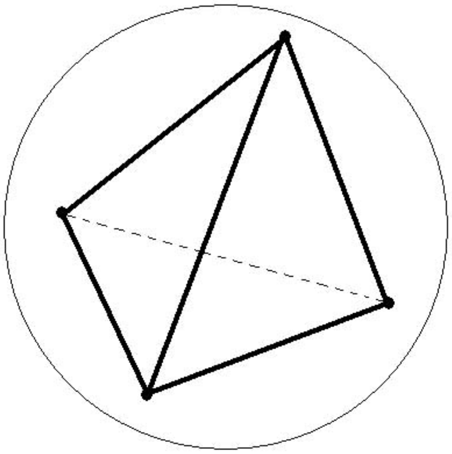
All of the Platonic solids (and only these) meet these rules, which make them "perfect:"

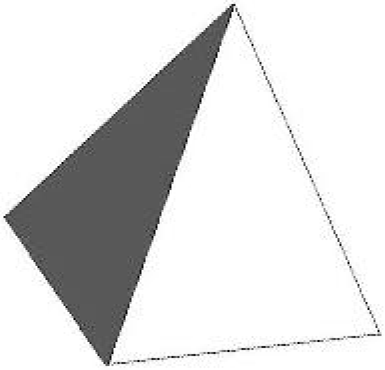
1. They look the same from any corner point.
2. For a given solid, each face is of the same shape.
3. Every edge and every angle is identical.



Just as the basic shapes can be derived from a circle, we start with a three-dimension circle to form the solids, that is, we start with a **sphere**. This is a circle that has been completely rotated about an axis, just like a globe of the earth. A sphere is also "perfect" in a certain way. Every point on it lies exactly the same distance from the center. Its surface is the smallest area that can be formed to enclose a given volume. This is why a droplet of liquid released in outer space naturally forms a near spherical shape; the surface tension forces the most compact enclosure for a given volume.

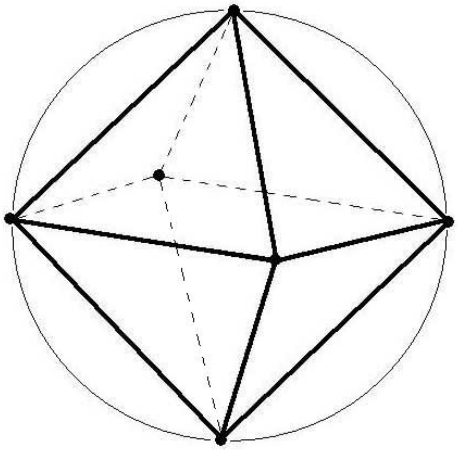
If we put <u>four points</u> on the surface of a sphere, each as far apart from each other as possible, we get a **tetrahedron**:

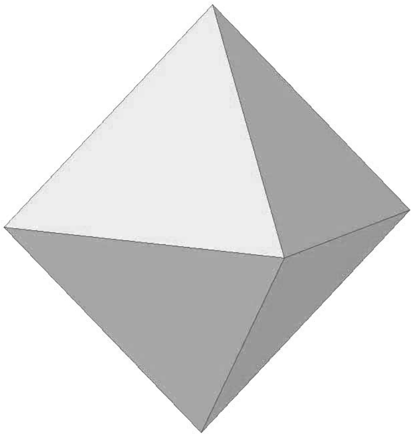




Every face of the tetrahedron is an equilateral triangle. The ancients thought that four elements made up world: fire, earth, air, and water. They associated each with one of the perfect solids. Plato associated the tetrahedron with fire. While a tetrahedron resembles a pyramid (Greek *pur* means fire, and the Greeks also call the tetrahedron *puramis*, from which we get the word pyramid), an Egyptian pyramid has a base that is square. But all faces of a tetrahedron are triangles, no face is square.

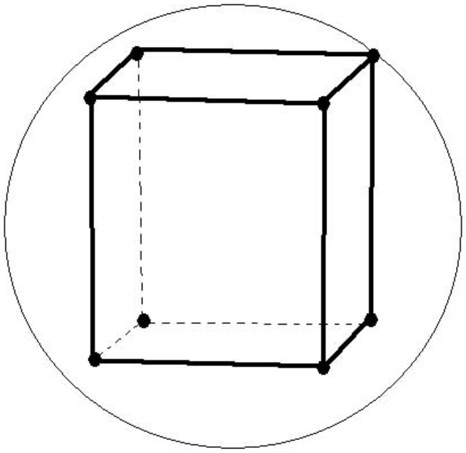
If we put <u>six points</u> on the surface of a sphere, each as far apart from each other as possible, we get an **octahedron**:

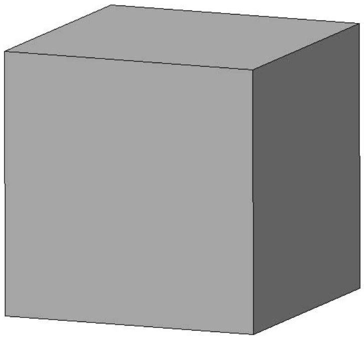




An octahedron has 8 faces. The top half of it is a pyramid. Octahedrons, like the other perfect solids, are found in minerals. Diamonds, for example, are often in this shape. Plato associated the octahedron with air.

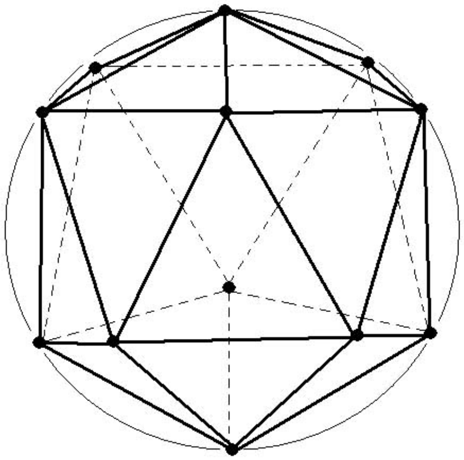
If we put <u>eight points</u> on the surface of a sphere, each as far apart from each other as possible, we get a **cube**:

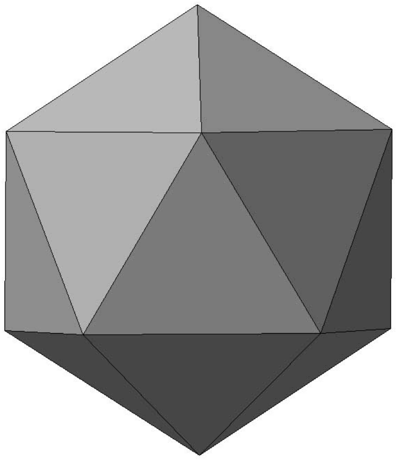




The cube was associated by Plato with earth, since its square base represents ultimate stability. But the cube has several qualities that the ancients regarded as indicative of a special significance. It has six "directions": north, south, east, west, up, and down. Numerology (attributing significance to various numbers) was important in the lives of some ancients, and 6 is the first "perfect" number since its factors 1, 2, and 3 all add up to the number itself. If you add up the number of the cube's edges (12), face diagonals (12), and internal diagonals (4) you get 28, which is the second perfect number (1 + 2 + 4 + 7 + 14 = 28). Various biblical and religious references to cubes are perhaps based on this special significance. Many molecular structures such as salt crystals and the arrangement of molecules follow a cube structure.

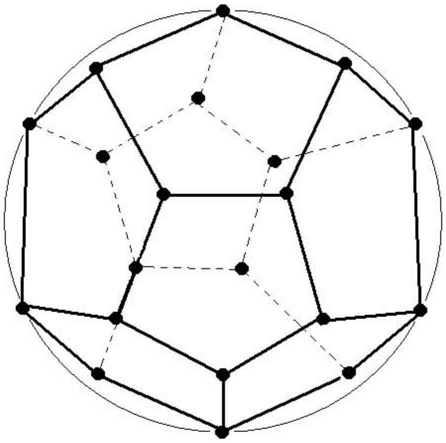
If we put twelve points on the surface of a sphere, each as far apart from each other as possible, we get an **icosahedron**:

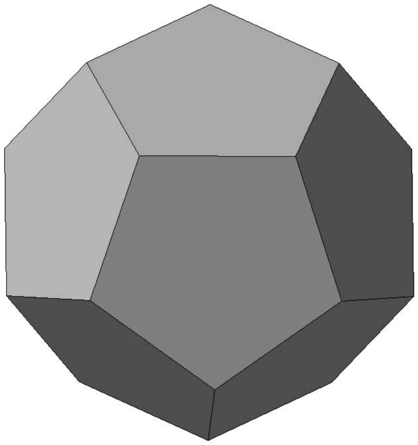




Three of the four elements recognized by the ancients were fluid: fire, air, and water. Water is the densest of the fluid elements, and Plato associated the icosahedron with water. If you join the two ends of any edge of the icosahedron to the center you form an isosceles triangle. As Daud Sutton notes in ***Platonic & Archimedean Solids*** this isosceles triangle has the same proportions as the four faces of the Great Pyramid of Eqypt. Is this mysterious? Probably not. The Egyptians, as did many other ancient peoples, knew about these solids and attributed significance to many aspects of them. Since they felt that God (or the Gods) had created these solids they believed that the solids could be used for guidance on how to conduct life and build structures that honored their deities.

Finally, if we put <u>twenty points</u> on the surface of a sphere, each as far apart from each other as possible, we get a **dodecahedron**:



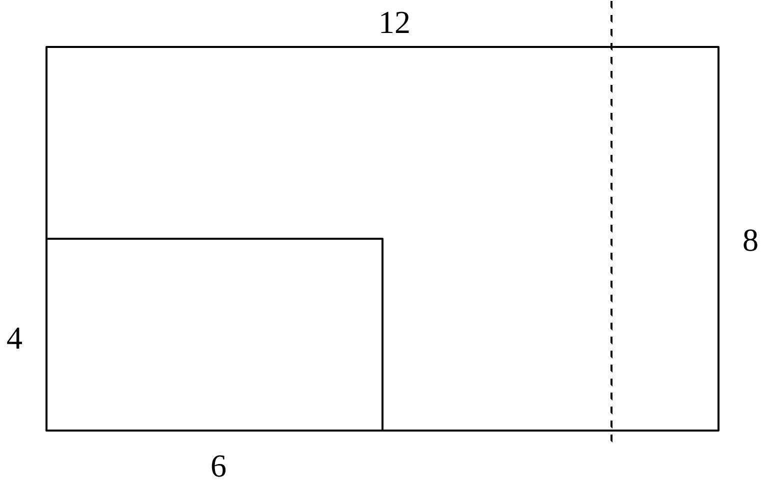


The dodecahedron has 12 faces which are pentagons, five-sided shapes. It was known as "the sphere of twelve pentagons." Plato associated it not with an earthly element but with the stars.

You can easily view and explore fold-outs of polyhedrons on a computer using a program named PolyPro. You can download PolyPro for free from **www.peda.com/polypro** on the web or access it on the CD/ROM associated with this workbook. You can use this trial version of the software and decide if you would like to register it with the originator to use it more extensively. The shaded illustrations of the perfect solids shown in this chapter were made using PolyPro (the line drawings were hand-made by the author).

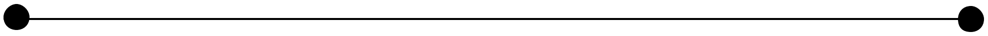
Proportions and the Golden Mean

A proportion is the relationship of two things with each other. For example, in a 4" x 6" photograph, the height is 4 inches and the length 6 inches (for a "landscape" orientation). The proportion is 4/6. If we enlarge the photo to an 8" x 12" size, the height and length are still in the same proportion.

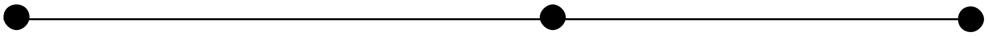


On the other hand, we can't enlarge a 4" x 6" to an 8" x 10" photo because they are not in the same proportion. The dashed line above shows you what would happen if you did this. Simply put, the "shape" of an 8 x 10 is not the same shape as a 4 x 6. (This is currently causing a lot of dismay as people fond of 4 x 6 snapshot prints make enlargements in the common 8 x 10 size!)

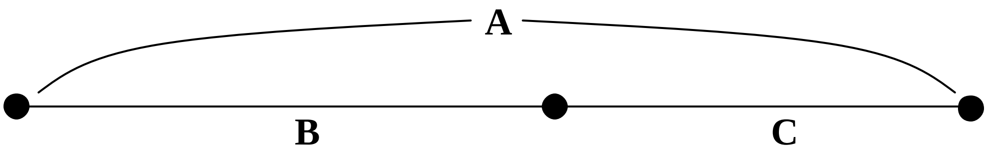
Now here is a puzzle the ancients were fond of: "Take a line. Divide it into two unequal parts such that the longer of the two segments is to the shorter as the whole line is to the longer segment." Whoa! How about saying that in plain words? Let's do it graphically. Here is a line:



Cut it into two by putting a point somewhere on it:



Now let's label the whole line and each segment so we can talk about these easily:



Place the dividing point so that A is to B as B is to C, that is A/B = B/C. **This can be true in only in one way!** This one way is called the "golden mean" and has many significant implications for art and architecture. At this point please visit the web site at this address: **http://goldennumber.net/neophite.htm**. Read the two pages at this location but also click on every hyperlink in the central section! Here is a golden rectangle. Can you see why it is so named? If B = 1, A = 1.618… (a never-ending decimal number).